Metastability for delayed differential equations

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In systems at phase transitions, two phases of the same substance may coexist for a long time before one of them dominates. We show that a similar phenomenon occurs in systems with delayed feedback, where short-term stable oscillatory patterns can also have very long lifetimes before vanishing into constant or periodic steady states. [S1063-651X(99)16211-1]

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I. INTRODUCTION

Metastability is often associated with phase transitions. For example, at the transition temperature, two phases of the same substance (for instance, liquid and solid at the wetting temperature) can coexist for an extremely long time before one of them eventually dominates. The slow evolution from spatial inhomogeneity to the homogeneous state results from the surface tension that tends to reduce the area of the interface between the two phases (e.g., [1,2]).

A simple equation that has been used as a model for the slow interface dynamics is

$$\partial_t u = \epsilon^2 \partial_x^2 u - \frac{dF}{du}(u), \tag{1}$$

where u represents the state variable of a substance at a point $x \in [0,1]$, and time t, F is a free energy function, and $\epsilon \ll 1$ measures the relative importance of the surface tension. It is assumed that F has two minima (see Fig. 1), at a and b, representing the two coexisting phases at the transition temperature where $F(a)=F(b)$. Notice that $u=a$ and $u=b$ are stationary solutions of Eq. (1). At the transition temperature Eq. (1) has solutions in which the coexistence of the two phases persist for an extremely long time (of the order of $e^{-1/\epsilon}$) before giving way to a homogeneous state where only one of the phases exists [2,3]. Solutions having these properties will be called metastable solutions. For simplicity, let us assume that $ab<0$. In terms of Eq. (1), a typical metastable solution is one that has a square wave shape with plateaus at a and b representing regions where the corresponding phase of the substance dominates, and the sign changes indicating the interfaces between the two phases. The slow evolution towards the homogeneous state translates into a slow decrease in the number of sign changes: as one phase is progressively absorbed by the other, consecutive sign changes merge together and vanish.

The goal of this paper is to show the existence of a similar metastability (slow evolution towards one of the equilibria) in systems with delayed feedback. To this end, we consider the prototype of such systems represented by the following delayed differential equations (DDEs):

$$\epsilon \dot{x}(t) = -x(t) + f(x(t-1)), \tag{2}$$

where f is essentially either an increasing function (positive feedback) or a decreasing function (negative feedback) and satisfies $f(0)=0$, and $|f'(0)|>1$.

Equation (2) appears as a model for many biological and physical systems such as nonlinear optical devices [4]. In the

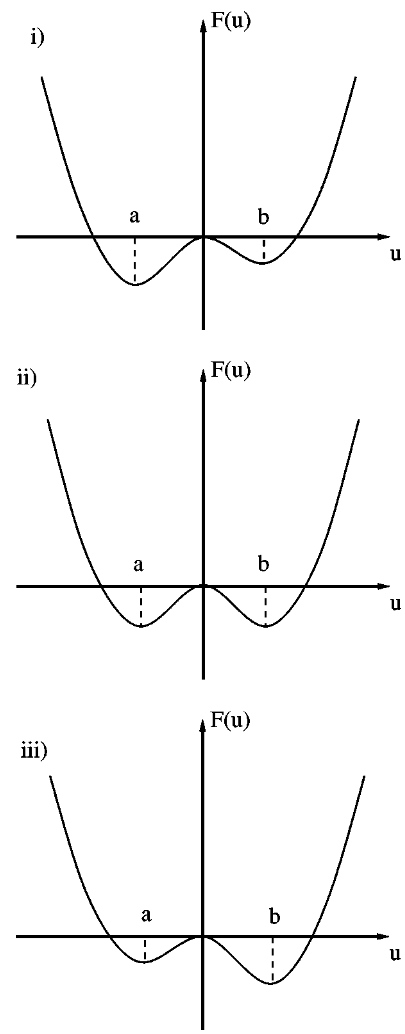

FIG. 1. Typical function $F(u)$ with two minima separated by a maximum: (i) at temperature lower than the transition temperature; (ii) at the transition temperature; (iii) at temperature higher than the transition temperature.

context of optical bistable devices the existence of very long transient "states" were already experimentally observed in the early 1980s [5], Appendix A, and [6]).

In analogy with Eq. (1), an oscillating solution of Eq. (2) is metastable if it changes slowly along time; that is, if the position and the number of its sign changes evolve slowly. Therefore, in order to show that Eq. (2) exhibits metastable solutions, and to determine the conditions for their existence, we need to study the evolution of the sign changes along oscillating solutions.

There is an important difference between the metastable patterns of the model for phase transitions described above and those we present in this paper. While the phase transition patterns are quasistatic with respect to evolution in time, ours are oscillatory and quasiperiodic. This means that there are initial conditions that after a fast transient lead to a periodic pattern that is short-time stable to perturbations and last for a very long time. The system seems to have achieved a steady periodic regime that, nevertheless, changes very slowly in time. Thus, in the case of positive feedback, for instance, the metastable pattern satisfies $x(t)=x(t+1+r\epsilon)$ up to exponentially small corrections in ϵ, where r is some constant to be determined in the following way. Substituting this approximation in Eq. (2), rescaling time as $t'=t\epsilon$, defining $y(t')=x(t)$, and taking the limit as $\epsilon\to 0$, we obtain the following advanced equation:

$$\dot{y}(t)=-y(t)+f(y(t+r)). \tag{3}$$

There are two values of r, $r=r_-$, $r=r_+$, for which Eq. (3) has solutions y_- and y_+ such that $y_+(-\infty)=a$, $y_+(+\infty)=b$, and $y_-(-\infty)=b$, $y_-(+\infty)=a$. Since y_+ and y_- make the transition between the equilibria a and b, Eq. (3) is called a transition layer equation. It is possible to show that the real solution of Eq. (2) can be approximated by sequences of functions y_+ and y_- conveniently rescaled and glued together. The metastability arises when f implies that $r_-=r_+\equiv r$. In the negative feedback case a similar analysis leads to a system of two advanced equations instead of the single Eq. (3), and it follows that there always exists metastability, in contrast to the case of positive feedback. The detailed discussion of both positive and negative feedback cases for smooth function f will be presented in a longer paper. Here we shall just consider two cases of piecewise constant f's [7]: $f(0)=0$, $f(x)=a$ for $x<0$, and $f(x)=b$ for $x>0$. The *positive feedback* case corresponds to $a<0<b$, and the *negative feedback* case corresponds to $a>0>b$.

A. Positive feedback

For this piecewise constant function f, positive and negative solutions tend monotonously to b and a, respectively. We are interested in the oscillating solutions; that is, those that display sign changes. The temporal evolution of these can be described through the propagation of the sign changes, that is, the consecutive zeros of the solutions can be computed iteratively.

Let us illustrate this for the solution x of Eq. (2) going through an initial condition ϕ satisfying $\phi(s)>0$ for $s<-\theta$, $\phi(s)<0$ for $s>-\theta$, and $\phi(0)=0$, for some θ in (0,1). A simple computation shows that x has two zeroes in the interval $(0,2-\theta)$: the first one at $t_1=1-\theta+\underline{\delta}$ with $\dot{x}(t)<0$, and the second one at $t_2=1+\overline{\delta}$ with $\dot{x}(t)>0$, where (see Fig. 2),

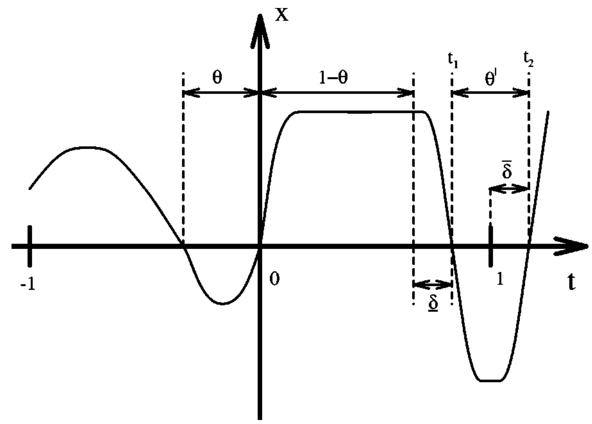

FIG. 2. Evolution of zeros for the positive feedback case.

$$\underline{\delta}=\epsilon\ln\left[\frac{b-a-be^{-(1-\theta)/\epsilon}}{-a}\right],$$

$$\overline{\delta}=\epsilon\ln\left[\frac{b-a-(b-a)e^{-\theta/\epsilon}-be^{-1/\epsilon}}{b}\right].$$

Then, the consecutive zeros of x are obtained by the iterations of the mapping $p:(0,1)\to\mathbb{R}$ given by

$$\theta'=p(\theta)=t_2-t_1=\theta+\overline{\delta}-\underline{\delta}. \tag{4}$$

For ϵ small $p(\theta)$ can be written as

$$p(\theta)=\theta+\epsilon\left[\ln\frac{|a|}{b}-e^{-\theta/\epsilon}+\frac{b}{b-a}e^{-(1-\theta)/\epsilon}\right]+R(\theta,\epsilon), \tag{5}$$

where if $\theta_m=\min\{\theta,1-\theta\}$ then $R(\theta,\epsilon)\exp(\theta_m/\epsilon)\to 0$ as $\epsilon\to 0$.

Using Eqs. (4) and (5), we can show that Eq. (2) has an unstable periodic orbit in which $\theta'=\theta\approx 1/2+\epsilon\ln[(b-a)/b]^{1/2}=\overline{\theta}$. If the initial θ is larger than $\overline{\theta}$ then its iterates slowly increase until they reach $\theta=1$. If the initial θ is smaller than $\overline{\theta}$ then its iterates slowly decrease until they reach $\theta=0$. The speed with which the zeros evolve toward either 0 or 1 depends on the relative positions of $|a|$ and b. When either $|a|>b$ or conversely $b>|a|$, $\theta'-\theta$ is of the order of ϵ. However, when $|a|=b$, Eq. (5) implies that $\theta'-\theta$ is of the order $(\epsilon)\exp(-\theta_m/\epsilon)$. In this case, provided that θ_m is larger than ϵ, the solution x of the initial condition ϕ with sign change at θ is a metastable square wave. This square wave subsists until the slow motion of θ towards 0 or 1 eventually accelerates when $\theta\approx\epsilon$, in which case the solution x is mostly equal to b, or $\theta\approx 1-\epsilon$, in which case the solution is mostly equal to a (this fact also has an analog with the phase transition metastability where "drops" of a nondominant phase are metastable provided they are sufficiently large).

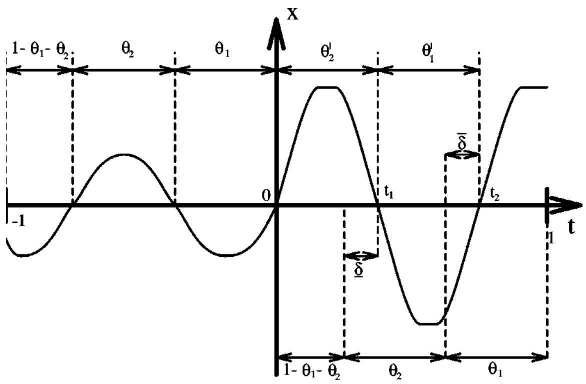

FIG. 3. Evolution of zeros for the negative feedback case.

The same analysis can be carried out in the case where the initial condition changes sign several times. It shows that the zeroes drift very slowly in a way similar to that described above.

Finally, notice that the condition $b/|a|=1$ is crucial for the existence of metastable patterns, otherwise $\theta'-\theta$ is of the order of ϵ. For the piecewise constant function *f*, a simple computation shows that r_- and r_+ of Eq. (3) are given by

$$r_+=\ln\left(1-\frac{a}{b}\right) \text{ and } r_-=\ln\left(1-\frac{b}{a}\right).$$

Therefore the metastability condition $r_+=r_-$ will be satisfied if and only if $|a|/b=1$.

B. Negative Feedback

The analysis of this case follows the same lines of the positive feedback case. The simplest type of initial condition that leads to a metastable pattern must have two sign changes. So let us assume that $\phi(s)<0$ for $s\in[-1,-\theta_1-\theta_2)\cup(-\theta_1,0)$, $\phi(s)>0$ for $s\in(-\theta_1-\theta_2,-\theta_1)$, and $\phi(0)=0$ (see Fig. 3). As in the positive feedback case, the solution *x* related to any initial function of this type can be easily computed. From this computation we obtain that *x* has only two zeroes in the interval (0,1): at $t_1=1-\theta_1-\theta_2+\underline{\delta}$, and at $t_2=1-\theta_1+\overline{\delta}$ (see Fig. 3), where

$$\underline{\delta}=\epsilon\ln\left[\frac{a-b+be^{-(1-\theta_1-\theta_2)/\epsilon}}{a}\right],$$

$$\overline{\delta}=\epsilon\ln\left[\frac{a-b-(a-b)e^{-\theta_2/\epsilon}-be^{-(1-\theta_1)/\epsilon)}}{-b}\right].$$

Defining $\theta_1'=t_2-t_1$ and $\theta_2'=t_1$ (see Fig. 3) the consecutive zeros of *x* are completely determined by iterations of the mapping $p:(0,1)\times(0,1)\to\mathbb{R}^2$ given by

$$\theta_1'=p_1(\theta_1,\theta_2)=\theta_2+\overline{\delta}-\underline{\delta},$$

$$\theta_2'=p_2(\theta_1,\theta_2)=1-\theta_1-\theta_2+\underline{\delta}.$$

Denoting (θ_1,θ_2) by θ we can write *p* as

$$\theta'=p(\theta)=A\theta+v+\epsilon\alpha+\epsilon\beta+R,$$

where

$$A=\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix},\quad \alpha=\begin{pmatrix}\ln[a/-b]\\ \ln[(a-b)/a]\end{pmatrix},$$

$$v=\begin{pmatrix}0\\1\end{pmatrix},\quad \beta=\begin{pmatrix}\left(\dfrac{-b}{a-b}\right)e^{-(1-\theta_1-\theta_2)/\epsilon}-e^{-\theta_2/\epsilon}\\ \left(\dfrac{b}{a-b}\right)e^{-(1-\theta_1-\theta_2)/\epsilon}\end{pmatrix},$$

and $R(\theta_1,\theta_2,\epsilon)\exp(\theta_m/\epsilon)\to 0$ as $\epsilon\to 0$ with $\theta_m=\min\{\theta_2,1-\theta_1-\theta_2\}$. Now, neglecting the exponentially small terms in *p* we define

$$\tilde{p}(\theta)=A\theta+v+\epsilon\alpha.$$

It is easy to verify that the third iterate of $\tilde{p}$ is the identity, therefore

$$p\circ p\circ p(\theta)=\theta+r(\theta,\epsilon),$$

where $\|r(\theta,\epsilon)\exp(\theta_m/\epsilon)\|<\text{const}$ for all $\epsilon\in[0,1]$. This implies the existence of metastable patterns provided that $\min\{\theta_1,\theta_2,1-\theta_1-\theta_2\}\gg\epsilon$. It is remarkable that in this case there is no condition on the coefficients a,b for the existence of metastability, in contrast to the positive feedback case.

II. CONCLUSION

Delayed differential equations support solutions with a large number of different patterns. There is numerical and experimental evidence (in the context of optical bistable devices) that many of them are stable [4–6,8]. It has been proposed by Ikeda and Matsumoto [8] (section 2.3) that these patterns could be potentially used to construct memory devices. The results in our paper suggest that the metastable patterns described above could be used to construct short memory devices. Initially a pattern with a certain number of signs would be given and this would be kept by the device for a very long, but finite, time. Depending on the size of ϵ this time can be large enough for practical applications.

In this paper we only considered the case of scalar delayed equations. We have shown that there is a big difference between positive and negative delayed feedback. Systems of delayed equations commonly appear as models of neural networks. They can have a mixture of positive and negative feedback loops. A numerical analysis of some simple networks showed that metastability can also occur for systems of delayed equations (Ref. [9]). Thus, it is an interesting open question, even in the case of piecewise constant nonlinearities, to determine conditions on systems of mixed positive-negative delayed feedback equations that ensure or preclude the existence of metastability. Depending on the answer to this question the potential for using delayed feedback networks to construct short memory devices can be greatly enhanced. The results in this paper also open room for many types of investigations that have been carried out in the context of phase transitions for some time, an example being the coarsening phenomenon [10].

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